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Solutions of Doubly and Higher Order Iterated Equations

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Michael Fisher once studied the solution of the equation $f(f(x))=x^2-2$. We offer solutions to the general equation $f(f(x))=h(x)$ in the form $f(x)=g(ag^{-1}(x))$ where a is in general a complex number. This leads to solving duplication formulas for $g(x)$. For the case $h(x)=x^2-2$, the solution is readily found, while the $h(x)=x^2+2$ case is challenging. The solution to these types of equations can be related to differential equations.

KEY WORDS: Dynamical systems; iterated equations; duplication equations; finite difference equations.

1. BACKGROUND ON ITERATED EQUATIONS

At a physics conference, during a coffee break, one of us (MFS), heard of a math problem that was supposed to test one's intelligence. The faster you could solve it, the smarter you were. It was remarked that Michael Fisher had solved it within five minutes. The problem was

$$f(f(x))=x^2-2 \tag{1}$$

This can be solve by writing

$$f(x)=g(ag^{-1}(x)) \tag{2}$$

Substituting this form for $f(x)$ one arrives at

$$f(f(x))=g(a^2g^{-1}(x))=x^2-2 \tag{3}$$

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Setting $a^2 = 2$ and letting $\theta = g^{-1}(x)$ we arrive at a double angle formula

$$g(2\theta) = x^2 - 2 = g(\theta)^2 - 2 \tag{4}$$

that is solved by inspection as

$$x = g(\theta) = 2\cos(\theta) \tag{5}$$

since $g(2\theta) = 2\cos(2\theta) = 4\cos^2(\theta) - 2$. (Note that if $f(x)$ solves a functional equation, then for any complex valued constant c, $f(c \cdot x)$ also solves this equation. As a result, both cos and cosh are solutions to the preceeding equation.) The solution for $f(x)$ can now be written as

$$f(x) = 2\cos(\sqrt{2}\cos^{-1}(x/2)) \tag{6}$$

In a similar manner, one can readily solve

$$f(f(x)) = 2x^2 - 1 \tag{7}$$

as

$$f(x) = \cos(\sqrt{2}\cos^{-1}(x)) \tag{8}$$

One of the authors, MFS, went home and remembered the problem as $f(f(x)) = x^2 + 2$, and after much effort could not make any progress. MFS passed the problem on to one of the other authors (ARB), who quickly realized that this equation had, in general, a complex non-analytic solution that has not yet been found in terms of known functions. ARB also quickly solved the simpler form Eq. (2). The authors then began a systematic study of multiple iterated equations that we report on here. As Michael Fisher figured prominently into our start on this topic, we dedicate this manuscript to him in honor of his 70th birthday.

As best as we can tell, it was Abel's who first introduced the concept of a functional equation. However, he gave no hint about why he was studying or introducing functional equations. In his paper,[4] the first line reads

"La fonction $f(x)$ étant donnée, trouver la fonction $\phi(x)$ par l'equation"

$$\phi(x) + 1 = \phi(f(x)) \tag{9}$$

which translates as "If the function $f(x)$ is given, find the function $\phi(x)$ from the equation:"

$$\phi(x) + 1 = \phi(f(x)) \tag{10}$$

Clearly, such equations look simple, but one quickly runs into questions of uniqueness, families of solutions, and questions of differentiability. For example $f(x+n)=f(x)$ admits an infinite number of solutions which includes all functions that are periodic in n. For another example, the seemingly simple equation

$$f(ax)=bf(x)-(b-1)\cos(x) \tag{11}$$

generates the nowhere differentiable Weierstrass function.

A particularly interesting point is the relation of iterated equations to finite difference equations. If $u_n = f(a^n)$ then in Eq. (10) we have $u_{n+1} = bu_n - (b-1)\cos(x)$.

As a first observation, all multiple iterated equations may be reduced to iterated equations (the analog of duplication equations), by the above method. Hence,we really need only treat iterated equations. While the iterated equation above may be solved by inspection, it begs for a more systematic treatment. A natural approach might be to reduce Eq. (4), to an ordinary differential equation. We now demonstrate that this is possible in this particular case.

Let

$$f(2x)=f(x)^2-2 \tag{12}$$

then

$$f'(2x)=f(x)\,f'(x) \tag{13}$$

squaring both equations we get

$$f(2x)^2=f(x)^4-4f(x)^2+4 \tag{14}$$

and

$$f'(2x)^2=f(x)^2\,f'(x)^2 \tag{15}$$

Subtract the value 4 from Eq. (14) and divide it into Eq. (15) to get

$$\frac{f'(2x)}{f(2x)^2-4}=\frac{f'(x)^2}{f(x)^2-4} \tag{16}$$

We conclude that both expression are constant and therefore

$$\frac{f'(x)^2}{f(x)^2-4}=c \tag{17}$$

and, using an initial condition to distinguish the cosine from the sine we arrive at the result $f(x)=2\cos(x)$.

However, we have cheated! There are two more problems to be cleared up. (1) Can we conclude that

$$\frac{f'(x)^2}{f(x)^2-4}=\text{constant} \tag{18}$$

And, if so, does the value of the constant matter?

The answer to the first question is no. The form of Eq. (16) is $g(ax) = g(x)$, a log periodic equation having an infinite number of solutions. One solution is $\cos(\omega\log(x))$ where ω is chosen properly. Any periodic function can be chosen in place of the cosine, and by adjusting ω we obtain a solution. If the solution of $g(ax)=g(x)$ is required to be analytic, then $g(x)=\text{constant}$. Assuming this, we obtain the solution $f(x)=2\cos(\omega x)$ as mentioned above. Once an analytic solution is found, an infinity of solutions can be obtained in the form $f(x)=2\cos(\alpha\cdot x\cdot\cos(\omega\log(x)))$, or more generally, $f(x)=2\cos(\omega\times g(x))$, where $g(2x)=g(x)$. In essence, in iterated equations, the log periodic functions behave as constants. Clearly, the answer to the second question is that the constant does not matter.

This example highlights an important class of functions that play a role in the solution of iterated equations, i.e., the log periodic functions. Given a log periodic function f and an arbitrary function g, then $g(f(x))$ is another log periodic function of the same period. Hence, even very simple iterated equations have infinitely many (non analytic) solutions.

A second class of important functions are those that satisfy the iterated equation: $f(az)=bf(z), b\neq 1$. A solution is $f(z)=z^{\alpha}$, so that $a^{\alpha}=b$. If we multiply this solution by any log periodic function of log period a, we obtain another solution.

$f(az)=bf(z)$ might be considered the second fundamental iterated equation, log periodic functions being the first. It is both simple and linear. Why this equation plays a fundamental role in the solution of iterated equations is this: If we are given the solution, f, of $f(2z)=f(z)^2+2$ then the solution, g, of $g(az)=g(z)^2+2$ is expressible as $g(z)=f(z^{\alpha})$, where $a^{\alpha}=2$. Thus, if we can find an analytic solution for some specific a, then we can find non analytic solutions for other values of a. Hence, all solutions (*a conjecture*) are expressible in terms of analytic functions and solutions of the two fundamental iterated equations.

2. ITERATED EQUATIONS AND DYNAMICAL SYSTEMS

The general first order, autonomous, replication equation is given by

$$f(ax) = G(f(x)) \tag{19}$$

the solution of this equation leads to a solution of the general nonlinear, autonomous, finite difference equation given by:

$$y_{k+1} = G(y_k) \tag{20}$$

If we are able to find a locally invertible solution of Eq. (19) then Eq. (20) is solved by

$$y_k = f(a^k f^{-1}(y_0)) \tag{21}$$

when f maps the complex plane into the domain of f^{-1}. Particular solutions then result from establishing the appropriate interval on which $f^{-1}(x)$ exists.

But regardless of whether a solution can be found, Eq. (21) provides direct insight into the mechanisms from which nonlinear equations arise, as well as insight into the basic mechanism of chaos. Clearly the value of a and the nature of the sequence of the powers of a tells us great deal about the potential dynamics of the solution of Eq. (20). A strategy for solving Eq. (20) now comes down to finding an a for which there is an analytic solution, f, of Eq. (19). The particular function f and the value of a then reveal the dynamics of Eq. (20).

3. ANALYTIC SOLUTIONS OF ITERATED EQUATIONS

The problem of solving Eq. (19) can be simplified by requiring that G be analytic. This covers a wide range of interesting cases.

If $f(a\,x) = G(f(x))$ and we assume that there exist a nonconstant analytic solution, we may *likely* (the general case is not solved as will be demonstrated by an example) obtain the terms of the Taylor series by direct computation. The process of obtaining the power series will require that the value of a (since not every value of a corresponds to an analytic solution) be fixed at some point and that $f(0)$, $f'(0)$ be fixed, also. Given these values, the remaining derivatives at 0 follow from differentiating the iterated equation. For example, when G is a polynomial, the possible values for $f(0)$ are obtained from the polynomial in $f(0)$:

$$f(0) = G(f(0)) \tag{22}$$

If λ_n are the roots of this equation, then $f(x) = \lambda_n$ are all solutions of the replication equation. As we are seeking locally invertible solutions, these solutions are discarded, hence some derivative must be nonzero and not infinite at 0. Given $f(0)$, we turn to the value of a and $f'(0)$ using

$$f'(ax)\,a = G'(f(x))\,f'(x) \tag{23}$$

Setting $x = 0$ we have

$$f'(0)\,a = G'(f(0))\,f'(0) \tag{24}$$

Possible solutions are $a = G'(f(0))$, and $f'(0) = 1$, or $f'(0) = 0$, thus deferring the determination of a until later. The choice $f'(0) = 1$ determines that the duplication constant a is $G'(f(0))$, which may be a complex number depending on the roots of $f(0) = G(f(0))$. If a is given in advance and $a \neq G(f(0))$, then $f'(0) = 0$. The second derivative equation is

$$f''(ax)\,a^2 = G''(f(x))\,f'(x)^2 + G'(f(x))\,f''(x)$$

For $x = 0$ we have

$$f''(0)(a^2 - G'(f(0))) = G''(f(0))\,f'(0) \tag{25}$$

Clearly, this process can be continued to obtain all Taylor coefficients. At each juncture, we have a choice of fixing a or fixing a derivative. Thus there are numerous possible solutions depending on a. By fixing a in advance, we may have only the roots of Eq. (22) as analytic solutions.

Once the formal Taylor series has been found, two problems remain. Find the radius of convergence of the series and find the domain of a local inverse of f. These theoretical questions are being investigated by one of the authors for her dissertation (BAB) and, thus, will not be discussed in detail here. The short answer is that there will generally be a positive radius of convergence when a Taylor series can be obtained, which may be the entire complex plane, and there will be a local inverse. We will illustrate these facts with some examples.

4. EXAMPLES

Example 1. Let $f(ax) = 2f(x)^2 - 1$, then $f(0) = 1, -1/2$. We choose $f(0) = 1$. Differentiating we get $af'(ax) = 2f(x)\,f'(x)$, and $af'(0) = 4f(0)\,f'(0)$. Choose $f'(0) = 1$, and then $a = 4$. Continuing, we get the power series for f:

$$f(x) = \sum_0^\infty b_k \frac{x^k}{k!}, \qquad 1/b_k = \prod_{j=1}^{k} (2j-1) \tag{26}$$

The first few terms of this series are

$$1+x+\frac{x^2}{(2!\cdot 1\cdot 3)}+\frac{x^3}{3!\cdot 1\cdot 3\cdot 5}\cdots \tag{27}$$

Clearly the series is uniformly convergent in the complex plane. If we choose $f'(0)=0$ and $f''(0)=-1$, then $a=2$ and we obtain the cosine series. Noninvertible solutions are the constant functions $f(x)=1$ and $f(x)=-1/2$, which also solve the associated nonlinear finite difference equation.

Example 2. Find a function g such that $g(g(x))=x^2+b$, where b is a constant.

First solve the replication equation for f.

$$f(ax)=f(x)^2+b \tag{28}$$

We expect that a will depend on b. We first note that for generating the derivative iterated equations, we need only consider $b=0$ as b enters into the formulae through $f(0)$. We write down the equations for five derivatives:

$$f'(ax)\,a-2f(x)\,f'(x)=0 \tag{29}$$

$$f''(ax)\,a^2-2f(x)\,f''(x)=2f'(x)^2 \tag{30}$$

$$f^{(3)}(ax)\,a^3-2f(x)\,f^{(3)}(x)=6f'(x)\,f''(x) \tag{31}$$

$$f^{(4)}(ax)\,a^4-2f(x)\,f^{(4)}(x)=8f'(x)\,f^{(3)}(x)+6f''(x)^2 \tag{32}$$

$$f^{(5)}(ax)\,a^5-2f(x)\,f^{(5)}(x)=10f'(x)\,f^{(4)}(x)+20f''(x)\,f^{(3)}(x) \tag{33}$$

We note that

$$f^{(n)}(ax)\,a^n-2f(x)\,f^{(n)}(x)=\sum_{j=1}^{N} c_j\,f^{(N-j)}(x)\,f^{(j)}(x) \tag{34}$$

where $N=n-1$. In Eq. (34), the coefficients c_j can be obtained from a polynomial P_N, where

$$\mathrm{P}_N(x)=\sum_{j=1}^{N} c_j\,x^j \tag{35}$$

which is obtained by iteration of the functional equation

$$\mathrm{S}(h(x)) = (x+1)\, h(x) + 2 \tag{36}$$

where $h(x)$ is an "initial condition" which is a function of x. In particular, choosing the initial function $h(x) = 2$

$$\mathrm{P}_N(x) = \mathrm{S}^{N-2}(2) \tag{37}$$

It is not clear yet that the functional Eq. (36) for the coefficients represents a simplification or an advantage in solving (28). Hopefully, the complexity in the solution of (28) is restated in (36) in a form that can be attacked by more sophisticated methods on infinite dimensional spaces.

For the special case $b = 0$, $f(x) = \exp(x)$ and

$$g(x) = f(\sqrt{2}\, f^{-1}(x)) = x^{\sqrt{2}}$$

Example 3. Find the solution to the mapping

$$x_{k+1} = \lambda\, x_k(1 - x_k), \qquad 1 < \lambda. \tag{38}$$

We seek a locally invertible analytic function satisfying the duplication equation

$$f(ax) = \lambda f(x)(1 - f(x)) \tag{39}$$

The polynomial equation for $f(0)$ gives $f(0) = 0, 1 - 1/\lambda$, and we exclude the initial condition $f(0) = 0$, the constant solution.

$$f'(ax)\, a + (2\lambda f(x) - \lambda)\, f'(x) = 0 \tag{40}$$

we may choose $f'(0) = 1$ and then $2\lambda f(0) - \lambda = -a$. Likewise we have

$$f''(0) = -\frac{2\,\lambda}{a^2 - a} \tag{41}$$

and

$$f^{(3)}(0) = \frac{3(2\lambda)^2}{(a^3 - a)(a^2 - a)} \tag{42}$$

5. GENERALIZATIONS

The equation

$$y_{k+1} = G(y_k) \tag{43}$$

is also solved by this method when G is not a polynomial but has all derivatives.

More importantly, the method extends to any number of dimensions. Let $X \in \mathbf{R}^n$ and $G: \mathbf{R}^n \to \mathbf{R}^n$ be C^∞. Then the solution of the duplication equation

$$F(A\,X) = G(F(X)) \tag{44}$$

where A is a Hermitian matrix provides the basis for solving the nth order nonlinear, autonomous, finite difference equation

$$X_{k+1} = G(X_k) \tag{45}$$

and the iterated equation

$$h^k(X) = G(X) \tag{46}$$

where h^k is h composed with itself k times. The solutions are

$$X_k = F(A^k F^{-1}(X_0)) \tag{47}$$

and

$$h(X) = F(A^{1/k} F^{-1}(X)) \tag{48}$$

respectively when F maps n-dimensional space into the domain of F^{-1}. Note that the radius of convergence and domain of the inverse are to be determined. In many cases, the matrix A may be taken to be a diagonal matrix D. Extensions to general linear spaces would require that the matrix A be replaced by a bounded linear operator, L, on a Banach space.

Example 4. The Hénon map is given by

$$G\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + y - cx^2 \\ dx \end{pmatrix} \tag{49}$$

Iteration of this equation leads to the finite difference equation

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 1 + y_k - cx_k^2 \\ dx_k \end{pmatrix} \tag{50}$$

We form the duplication equation with a diagonal matrix:

$$F\begin{pmatrix} ax \\ by \end{pmatrix} = \begin{pmatrix} f(ax, by) \\ g(ax, by) \end{pmatrix} = G(F(X)) \tag{51}$$

where

$$G(F(X)) = \begin{pmatrix} 1+g(x,y)-cf(x,y)^2 \\ df(x,y) \end{pmatrix} \tag{52}$$

Extending the ideas of the first-order case we solve for $f(0,0)$, $g(0,0)$ using the duplication equation. We proceed to solve for the Taylor coefficients for f, g by the same method as in the first-order case, using partial derivatives instead of ordinary derivatives. Doing this we see that $g(0,0) = df(0,0)$ and $cf(0,0)^2 - 2f(0,0) + 1 = 0$. As in the first-order case, $f(0,0)$ may have multiple solutions. Next, we differentiate f, g partially with respect to x, y to get

$$f_x(ax, by) = g_x(x,y) - 2\,cf(x,y)\,f_x(x,y) \tag{53}$$

leading to

$$af_x(0,0) = g_x(0,0) - 2\,cf(0,0)\,f_x(0,0) \tag{54}$$

$$bf_y(0,0) = g_y(0,0) - 2\,cf(0,0)\,f_y(0,0) \tag{55}$$

$$ag_x(0,0) = df(0,0) \tag{56}$$

$$bg_y(0,0) = df(0,0) \tag{57}$$

continuing, we derive all necessary partial derivatives. In the event that A must be chosen as a nondiagnonal matrix to assure the existence of a locally invertible solution, the computation of derivatives becomes more tedious, but poses no additional theoretical problems beyond the first-order one-dimensional case.

Once this has been completed and we have obtained the map F, we have the result

$$H^k(X_0) = F(A^k F^{-1}(X_0)) \tag{58}$$

Where $H(X)$ is the Hénon map. Dropping subscripts we get the factorization of the Hénon map:

$$H^k(X) = F(A^k F^{-1}(X)) \tag{59}$$

This equation states that H is conjugate (i.e., has the same dynamical properties) to the map defined by A on the appropriate subspace of $\mathbf{R}^2$. The differentiable conjugacy is given by F. If the matrix A is hyperbolic (has one eigenvalue greater than one and one less than one), F maps two-dimensional space onto a bounded subset, and F^{-1} exists over a sufficient range, then we have proof that H is chaotic using only vector calculus.

6. POST SCRIPT

Considerably more can be said about these equations and their connection to the theory of finite difference equations, and nonlinear dynamics generally. BAB is conducting research in this area. With regard to the errant equation, $g(g(x)) = x^2 + 2$, the associated finite difference equation is known to be conjugate to the exponential function for large x. Further research is in progress. With regard to the Taylor series question noted earlier, the equation $f(ax) = f(x)\exp(f(x))$ is analytic, but whether f has any analytic solution is not known.

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